An Introduction to Donaldson's Polynomial Invariants

Keshu Zhou
Department of Mathematics of USTC

Abstract

This note is a brief introduction to Donaldson's polynomial invariants(cf [6]). We will begin with basic facts about instanton moduli spaces and construct these invariants via intersection theory. Finally we will discuss about an explicit example and give its application.

Contents

1 Instanton moduli spaces

We start with the derivation of instanton moduli spaces and some fundamental properties. Given a closed Riemannian 4-manifold X and an Hermitian bundle E over X, we say a connection A on E is anti-self-dual if its curvature satisfies $*F_A = -F_A$. Here $*$ is the Hodge star operator given by underlying Riemannian metric. Such connections are called instantons. Clearly ASD condition is preserved by gauge group action, so we consider the set of all instantons moduli gauge action, which is the moduli space of instantons. The fundamental fact is the following theorem(cf [2]):

Theorem 1.1 (**Freed-Uhlenbeck,1984**). *Fix a non-trivial $SU(2)$-bundle E over a simply-connected 4-manifold X with second Chern number k. If $b^+(X) > 0$, then for generic smooth Riemannian metric g of X, the moduli space $M_k(g)$ contains no reducible connections and is a smooth manifold of dimension $8k - 3(b^+ + 1)$.*

Orientability of moduli spaces. This is a general fact for all 4-manifolds, not only just simply-connected case. Let $E \to X$ be an $SU(2)$-bundle, and let M^s be the open subset of regular irreducible connections, then Donaldson proved in [4]:

Theorem 1.2 (**Donaldson,1987**). *The moduli space M^s is orientable, with a canonical orientation induced by an orientation of the space $H^1(X) \oplus H^+(X)$. Here $H^+(X)$ denotes the maximal positive subspace of $H^2(X)$ with respect to the intersection form.*

Compactification of moduli spaces. The compactification is defined by Donaldson in [3], essentially due to Uhlenbeck. Donaldson defined the ideal connection to be a pair $([A], (x_1, ..., x_l))$ where $[A]$ is a point in M_{k-l}, with its curvature density defined to be the measure $|F_A|^2 + 8\pi^2 \sum_j \delta_{x_j}$. We say a sequence of ideal connections converges weakly if their curvature densities converge as measures and their connections converge in C^∞ over punctured manifold. The notion of weak convergence endows the set of all ideal ASD connections

$$IM_k = M_k \cup M_{k-1} \times X \cup M_{k-2} \times s^2(X) \cup ... \cup M_0 \times s^k(X)$$

with a topology. Here $s^l(X)$ denotes the l-th symmetric product of X. Denote by $\overline{M}_k$ the closure of M_k in IM_k. The main result is the following:

Theorem 1.3 (**Donaldson,1986**). *The space $\overline{M}_k$ is compact.*

2 Construction of Donaldson's invariants

We are going to construct Donaldson's polynomial invariant(cf [5]). Let $\mathscr{B}^*_X$ be the orbit space of irreducible connections on E over X. View the moduli space M as a subset of $\mathscr{B}^*_X$. We wish to define invariants of X via a pairing $\langle \beta, [M] \rangle$, for certain cohomology classes $\beta \in H^*(\mathscr{B}_X^*)$. But usually M is non-compact, so we need to modify the definition via intersection theory.

Assume now the dimension of moduli space M_k is even, say dim $M_k = 2d$, where $d = d(k) = 4k - \frac{3}{2}(b^+(X) + 1)$. This requires that b^+ is odd, so we may assume $b^+(X)$ is odd and ≥ 3 to avoid reducible connections(cf Theorem 1.1). Fix an orientation Ω for $H^+(X)$, we seek for invariants $q_{k,\Omega}(\Sigma_1, ..., \Sigma_d)$ satisfying the following property:

(i) $q_{k,\Omega}(\Sigma_1, ..., \Sigma_d)$ depends on Σ_i only through its homology class;

(ii) $q_{k,\Omega}(\Sigma_1, ..., \Sigma_d)$ is multilinear and symmetric in $[\Sigma_i]$;

(iii) $q_{k,-\Omega} = -q_{k,\Omega}$;

(iv) q_k is natural, in the sense that if $f : X \to Y$ is an orientation-preservation diffeomorphism, then

$$q_{k,f^*\Omega}(f_*(\Sigma_1), ..., f_*(\Sigma_d)) = q_{k,\Omega}(\Sigma_1, ..., \Sigma_d).$$

where Σ_i are some embedded surfaces of X in general position.

Determinant line bundles. Suppose we have a family of operators $\{D_t : \Gamma(E_t \otimes V) \to \Gamma(E_t \otimes W)\}$ parametrized by a space T, we construct a line bundle over T, whose fibres are:

$$\det \text{ind}(D_t) = \Lambda^{\max}(\text{Ker } D_t) \otimes (\Lambda^{\max}(\text{Coker } D_t))^*$$

In our case, choose a square root of the canonical bundle K of a Riemann surface Σ, i.e. a line bundle $K^{\frac{1}{2}}$ with $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} = K$. The Dirac operator then becomes the usual $\overline{\partial}$ operator twisted by $K^{\frac{1}{2}}$, written as $\delta_\Sigma : \Omega^{0,0} \otimes K^{\frac{1}{2}} \to \Omega^{0,1} \otimes K^{\frac{1}{2}}$. We define $\mathscr{L}_\Sigma$ to be the determinant line bundle over $\mathscr{B}_\Sigma^*$ for the family of operators $\delta_{\Sigma,A}^*$, obtained by coupling δ_Σ^* to the connection A. For any tubular neighbourhood $\nu(\Sigma)$ of Σ, we also write $\mathscr{L}_\Sigma \to \mathscr{B}_{\nu(\Sigma)}^*$ to be the pull-back of previous line bundle via the restriction map.

Transversal intersections. Now suppose the tubular neighbourhoods $\nu(\Sigma_i)$ are sufficiently small such that for distinct i, j, k:

$$\nu(\Sigma_i) \cap \nu(\Sigma_j) \cap \nu(\Sigma_k) = \varnothing$$

For each i we choose a section s_i of the line bundle $\mathscr{L}_{\Sigma_i}$ over $\mathscr{B}_{\nu(\Sigma_i)}^*$ and let V_{Σ_i} be its zero-set. Transversality argument shows:

Proposition 2.1. *We can choose sections s_i such that the intersection $M_k \cap V_{\Sigma_1} \cap ... \cap V_{\Sigma_d}$ is transverse. Moreover, the intersection is compact, and hence finite. Here the precise meaning of the intersection is that $M_k \cap V_\Sigma = \{[A] \in M_k | [A|_{\nu(\Sigma)}] \in V_\Sigma\}$.*

We are now in a position to define Donaldson's polynomial invariants. Each point of the intersection $M_k \cap V_{\Sigma_1} \cap ... \cap V_{\Sigma_d}$ carries a sign ± 1 since both M_k and the normal bundles of V_{Σ_i} are oriented. We define $q_{k,\Omega}$ by counting these points with signs. The main assertion is that the definition is independent of all the choices we have made:

Theorem 2.2 (Donaldson,1990). *Under our assumption, if moreover k is in the stable range, i.e. $d(k) \geq 2k+1$, then the integer $q_{k,\Omega}$ satisfies properties (i)-(iv) and is independent of the choices made as well as the underlying Riemannian metric, thus defining a differential topological invariant.*

Finally we want to point out that parallel theory holds when the gauge group is $SO(3)$ with a little bit more careful treatment.

3 Application to differential topology

It turns out that Donaldson's invariants are unstable under the connected sum operation, thus being plausible to obtain new information beyond the intersection form. In the following part, we will consider a K3 surface example to deduce the failure of h-cobordism theorem in dimension 4, thus showing that these invariants are highly non-trivial.

Recall a K3 surface is a compact simply-connected complex surface with trivial canonical bundle. Let X be a K3 surface and let $\alpha \in H^2(X;\mathbb{Z}_2)$ be a class with $\alpha^2 = 2$ (mod4). Consider the $SO(3)$-bundle F with $w_2(F) = \alpha$ and $p_1(F) = -6$. Under such condition, the formal dimension of the moduli space M_F is zero, so in this case the polynomial invariant is just counting with signs the points in the moduli space. Using algebraic geometry argument one can show that M_F contains exactly one point(see [5] or [6]), or equivalently:

Proposition 3.1. *There is a class α and orientation Ω such that $q(\alpha, \Omega) = 1$.*

An immediate corollary is the following:

Corollary 3.2. *There is no diffeomorphism of X which acts trivially on $H^2(X;\mathbb{Z}_2)$ but reverses the orientation of $H^+(X)$.*

Proof. Let $f : X \to X$ be any orientation-preserving diffeomorphism, then due to the naturality of the invariant we see:

$$q(f^*(\alpha), f^*(\Omega)) = q(\alpha, \Omega)$$

If $f^*(\alpha) = \alpha$ but $f^*(\Omega) = -\Omega$, we will see that $q(\alpha, \Omega) = 0$, contrary to Proposition 3.1. □

Now we can show the h-cobordism theorem fails in dimension 4.

Corollary 3.3. *There is a simply-connected 5-dimensional h-cobordism which is not a product.*

Proof. Any h-cobordism W between simply-connected 4-manifolds X_1 and X_2 induces an isomorphism $f_W : H^2(X_1) \to H^2(X_2)$, preserving the intersection forms. The converse is also true due to Wall(cf [1]), i.e. any form-preserving isomorphism between the cohomology of simply-connected 4-manifolds can be realized as f_W for some h-cobordism W.

Now take X_1 and X_2 both to be the K3 surface and obtain an h-cobordism W realizing the map -1 on H^2. This cannot be a product, otherwise we obtain a diffeomorphism of X realizing -1 on cohomology, which contradicts with Corollary 3.2 since $H^+(X)$ has odd dimension. □

Acknowledgements

I am very grateful to my supervisor Associate Professor Bin Xu. He introduced me into this beautiful and exciting area of mathematics and spent a lot of time organizing our seminar. I also want to express my gratitude to Professor Di Yang and my classmates in Hua Loo-Keng Seminar. I learned a lot from all of you in this class. Finally I want to thank my friend Tianyang Sun for fixing some problems when dealing with LaTeX.

References

[1] C.T.C. Wall: On simply-connected 4-manifolds. Journal of the London Mathematical Society **39**, 141-49(1964).

[2] D.S. Freed and K.K. Uhlenbeck: *Instantons and Four-Manifolds.* Springer-Verlag New York Inc., 1984.

[3] S.K. Donaldson: Connections, cohomology and the intersection forms of 4-manifolds. J. Diff. Geom. **24**, 275-341(1986).

[4] —: The orientation of Yang-Mills moduli spaces and 4-manifold topology. J. Diff. Geom. **26**, 397-428(1987).

[5] —: Polynomial invariants for smooth 4-manifolds. Topology **29**, 257-315(1990).

[6] — and P.B. Kronheimer: *The Geometry of Four-Manifolds.* Clarendon Press, Oxford, 1990.